

A SHORT REMARK ON THE POLARON IN THE SEMI-RELATIVISTIC PAULI-FIERZ MODEL

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ABSTRACT. We consider the polaron of the spinless semi-relativistic Pauli-Fierz model. The Hamiltonian of the model is defined by $H(\mathbf{P}) = \sqrt{(\mathbf{P} - d\Gamma(\mathbf{k}) + e\mathbf{A})^2 + M^2} + d\Gamma(\omega_m)$, where $\mathbf{P} \in \mathbb{R}^3$ is the momentum of the polaron, $d\Gamma(\cdot)$ denotes the second quantization operator and $\omega_m = |\mathbf{k}| + m$ denotes the dispersion relation of the photon with virtual mass $m \geq 0$. Let $E(\mathbf{P})$ be the lowest energy of $H(\mathbf{P})$. In this paper, we prove the inequality

$$E(\mathbf{P} - \mathbf{k}) - E(\mathbf{P}) + \omega_m(\mathbf{k}) \geq m,$$

for all $\mathbf{P}, \mathbf{k} \in \mathbb{R}^3$.

1. DEFINITION AND MAIN RESULT

We consider a system of a charged spinless relativistic particle interacting with the quantized radiation field with fixed total momentum $\mathbf{P} \in \mathbb{R}^3$. The Hilbert space of the model is the Fock space

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_{\text{sym}}^n L^2(\mathbb{R}^3 \times \{1, 2\}) \right] \quad (1)$$

with $\bigotimes_{\text{sym}}^0 L^2(\mathbb{R}^3 \times \{1, 2\}) =: \mathbb{C}$. Let $d\Gamma(\cdot)$ be the second quantization operator. The Hamiltonian of the model is defined by

$$H(\mathbf{P}) := \sqrt{(\mathbf{P} - d\Gamma(\mathbf{k}) + e\mathbf{A})^2 + M^2} + d\Gamma(\omega_m), \quad (2)$$

where e is the coupling constant, $M > 0$ is the mass of the particle, $\omega_m(\mathbf{k}) = |\mathbf{k}| + m$ is the photon dispersion relation and \mathbf{A} is the quantized magnetic vector potential at the origin $\mathbf{x} = 0$. See [2] for more detailed definition. The self-adjointness of $H(\mathbf{P})$ was studied in [2] and [1, Corollary 7.62]. We assume the following

[H.1] There exist a dense domain $\mathcal{D} \subset \mathcal{F}$ such that, for all $\mathbf{P} \in \mathbb{R}^3$,

$$\mathcal{D} \subset \text{Dom}((\mathbf{P} - d\Gamma(\mathbf{k}) + e\mathbf{A})^2) \cap \text{Dom}(d\Gamma(\omega_m)) \quad (3)$$

and $H(\mathbf{P})$ is essentially self-adjoint on \mathcal{D} .

Clearly, $H(\mathbf{P})$ is bounded from below, and we define the ground state energy:

$$E(\mathbf{P}) := \inf \text{spec}(H(\mathbf{P})) \quad (4)$$

The main theorem of this paper is the following:

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Theorem 1.1. *For all $m \geq 0$, the inequality*

$$E(\mathbf{P} - \mathbf{k}) - E(\mathbf{P}) + \omega_m(\mathbf{k}) \geq m, \quad \mathbf{k}, \mathbf{P} \in \mathbb{R}^3 \quad (5)$$

holds.

Remark 1.2. We define

$$\Delta(\mathbf{P}) := \inf_{\mathbf{k} \in \mathbb{R}^3} \{E(\mathbf{P} - \mathbf{k}) - E(\mathbf{P}) + \omega_m(\mathbf{k})\} \quad (6)$$

As discussed in [2], $\Delta(\mathbf{P})$ is the spectral gap at the lowest energy of $H(\mathbf{P})$. The inequality (5) implies that the spectral gap is open uniformly in \mathbf{P} . To establish (5), our photon dispersion relation $|\mathbf{k}| + m$ was important. Similar result may not hold for the standard massive dispersion relation $\omega_m(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$.

Remark 1.3. This is remarkable that (5) holds for all \mathbf{P} and there is no restriction on the other parameters. This fact is different from the non-relativistic QED where the spectral gap is open only for $\mathbf{P}^2/2M < 1$. The uniform spectral gap is a nature of the relativistic dynamics.

Remark 1.4. To prove (5), we only use the operator monotonicity. Unfortunately, our method works only for the spinless case.

Remark 1.5. It is strongly expected that $H(\mathbf{P})$ with $m = 0$ has ground state for all $\mathbf{P} \in \mathbb{R}^3$ and $e \in \mathbb{R}$ under suitable conditions including the infrared regularization. By (5), the massive Hamiltonian $H(\mathbf{P})$, ($m > 0$) has a ground state for all $m > 0$. But, in our opinion, it is difficult to construct a ground state of massless model as the massless limit $m \downarrow 0$.

2. PROOF OF THEOREM 1.1

By the variational principle, (5) follows from the operator inequality

$$H(\mathbf{P} - \mathbf{k}) + \omega_m(\mathbf{k}) \geq m + H(\mathbf{P}), \quad (7)$$

on \mathcal{D} . We set

$$K(\mathbf{P}) := \sqrt{(\mathbf{P} - d\Gamma(\mathbf{k}) + e\mathbf{A})^2 + M^2}. \quad (8)$$

Then (7) is equivalent to

$$K(\mathbf{P} - \mathbf{k}) \geq K(\mathbf{P}) - |\mathbf{k}|, \quad (9)$$

on \mathcal{D} . By Löwner-Heinz inequality, (9) follows from

$$K(\mathbf{P} - \mathbf{k})^2 \geq (K(\mathbf{P}) - |\mathbf{k}|)^2, \quad (10)$$

in the sense of the quadratic forms on \mathcal{D} . By expanding the square of the operator, we know that (10) is equivalent to

$$|\mathbf{k}| \cdot K(\mathbf{P}) \geq \mathbf{k} \cdot (\mathbf{P} - d\Gamma(\mathbf{k}) + e\mathbf{A}). \quad (11)$$

By the Löwner-Heinz inequality, (11) follows from

$$\mathbf{k}^2 K(\mathbf{p})^2 \geq [\mathbf{k} \cdot (\mathbf{P} - d\Gamma(\mathbf{k}) + e\mathbf{A})]^2 \quad (12)$$

We set $\mathbf{B} = (B_1, B_2, B_3) := \mathbf{P} - d\Gamma(\mathbf{k}) + e\mathbf{A}$. Note that, for all self-adjoint operators a, b , it holds that $ab + ba \leq a^2 + b^2$ in the sense of the quadratic forms on $\text{Dom}(a) \cap \text{Dom}(b)$. Then we have

$$\left(\sum_{j=1}^3 k_j B_j \right)^2 = \sum_{j,l=1}^3 k_j B_j k_l B_l \quad (13)$$

$$\leq \frac{1}{2} \sum_{j,l} (k_j^2 B_j^2 + k_l^2 B_l^2) \quad (14)$$

$$= \sum_j k_j^2 \sum_l B_l^2 = |\mathbf{k}|^2 \cdot \mathbf{B}^2 \quad (15)$$

$$\leq |\mathbf{k}|^2 \cdot (\mathbf{B}^2 + M^2) \quad (16)$$

on \mathcal{D} . Therefore (12) holds.

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